MATH 122B: TAKE HOME FINAL

Any theorems covered during lecture may be referenced. Any other theorems or nontrivial claims must be provided with proof. A claim is nontrivial if you do not know why its true., i.e. no "From Theorem 2 of Book X".

Due in class on Thursday or online Friday

(1) Show that $\displaystyle\int_{S_N} \frac{dz}{z^3 \cos(\pi z)} \to 0$ as $N \to \infty$, where S_N is the square $[-N, N] \times [-iN, iN]$.

Use this to compute $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$.

(2) Let f be holomorphic on $\mathbb{C}$ and define the function

$$M(f, r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta.$$

Show that M is strictly increasing in r, unless f is a constant. Hint: Consider the series expansion of f centered at 0.

Using the result, find all entire (holomorphic on $\mathbb{C}$) functions such that

$$\iint_{\mathbb{C}} |f(z)|^2 dA < \infty$$

(3) Compute the integral $\displaystyle\int_0^{\infty} \frac{dx}{x^a + 1}$ for any real $a > 1$

(4) Let F be holomorphic in $|z| < 1 + \varepsilon$ for some $\varepsilon > 0$, and maps the closed unit disk into the open unit disk. Show that F has a fixed point, i.e. there is a z in the open unit disk such that $F(z) = z$. Hint: Rouché's theorem with $f(z) = -2z$ and $g(z) = F(z) + z$.

(5) Assume that f is holomorphic and one-to-one in D, hence an inverse f^{-1} exists. Show that

$$g(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{sf'(s)}{f(s) - z} ds$$

where γ is a closed simple contour, defines the inverse of f inside γ,i.e., show that $g(f(z_0)) = z_0$. In particular, this would show that f^{-1} is holomorphic. Note we cannot use the Cauchy integral formula on f^{-1} directly since we have not shown that f^{-1} is holomorphic yet, hence you would have to show that fact in another method before applying Cauchy integral formula. Consider multiplying and dividing by $z - z_0$.

(6) Show that any open disk is conformally equivalent to any open half-plane.

(7) Let n be a positive integer. Prove that $\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta) d\theta = \frac{2\pi}{n!}$

Solution

(1) We can show that $\tan(\pi z)$ is bounded on S_N by the same method that we used in lecture. Since $\sec^2(\pi z) = 1 + \tan^2(\pi z)$, $\sec(\pi z)$ is bounded on S_N. Then

$$\int_{N\to\infty} \frac{dz}{z^3\cos(\pi z)} \leq \frac{M}{N^2} \to 0$$

as $N \to \infty$, where M is some constant. By residue theorem,

$$\frac{1}{2\pi i}\int_{S_N} \frac{dz}{z^3\cos(\pi z)} = \sum_{n=-\infty}^{\infty} \mathrm{Res}(\frac{1}{z^3\cos(\pi z)}, \frac{2n+1}{2}) + \mathrm{Res}(\frac{1}{z^3, \cos(\pi z)}, 0)$$

Now

$$\mathrm{Res}(\frac{1}{z^3\cos(\pi z)}, \frac{2n+1}{2}) = \frac{8(-1)^n}{\pi(2n+1)^3}$$
$$\mathrm{Res}(\frac{1}{z^3\cos(\pi z)}, 0) = \frac{\pi^2}{2}.$$

We can split the sum as

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \sum_{-\infty}^{0} \frac{(-1)^n}{(2n+1)^3} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^3}$$
$$= 2\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^3}$$

which leads to the desired sum.

(2) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then

$$|f(re^{i\theta})|^2 = \sum a_n r^n e^{in\theta}\overline{\sum a_m r^m e^{-im\theta}}.$$

Since $\int_0^{2\pi} e^{in\theta} = 0$ for $n \neq 0$, the only terms that are nonzero after integrating $|f|^2$ are when $n = m$, hence

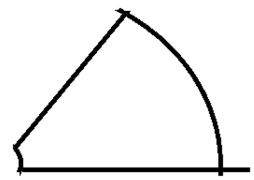
$$M(f,r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta$$
$$= \sum_n |a_n|^2 r^{2n}$$

which is a sum of nonzero terms, hence must increase with respect to r. Now

$$\int_0^{\infty} M(f,r) r dr = \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} |f(re^{i\theta})|^2 r d\theta dr$$
$$= \iint_{\mathbb{C}} |f|^2 dA < \infty$$

Since we showed that M is increasing with respect to r, the only way the integral can be finite is if $M = 0$. However, $M = 0$ implies that each term $a_n = 0$, hence this is the series expansion for $f \equiv 0$, which is the only possibility.

(3) We use the contour



Where the angle is $\frac{2\pi i}{a}$. We need to exclude 0 if a is not a positive integer since it would be a branch point. The only singularity for the function $\frac{1}{z^a+1}$ is at $z = e^{i\pi}a$. Now the circular components are easily shown to vanish for $R \to \infty$ and $r \to 0$. The slanted line contour is parametrized by $z = xe^{i2\pi/a}$, x from R to r so

$$\int_C \frac{dz}{z^a+1} = -e^{2\pi i/a} \int_r^R \frac{dx}{x^a+1}$$

Hence

$$(1 - e^{2\pi i/a}) \int_0^\infty \frac{dx}{x^a+1} = 2\pi i \operatorname{Res}(\frac{1}{z^a+1}, e^{i\pi}a)$$

which leads to

$$\int_0^\infty \frac{dx}{x^a+1} = \frac{\pi}{a\sin(\pi/a)}.$$

(4) We want to show that $F(z) - z$ has exactly one zero in the open unit disk. The condition that F maps the closed unit disk into the open unit disk means that $|F(z)| < 1$ for $|z| \leq 1$ and in particular for $|z| = 1$. Applying Rouché's theorem to the given pair, $|f(z)| = 2|z| = 2$ on $|z| = 1$ and

$$|g(z)| = |F(z) + z| \leq |F(z)| + |z| < 2 = |f(z)|$$

hence $F(z) - z$ has the same number of zeroes as $-2z$, which is exactly one.

(5) We want to show that $g(f(z_0)) = z_0$. Now

$$\begin{aligned} g(f(z_0)) &= \frac{1}{2\pi i} \int_\gamma \frac{sf'(s)}{f(s) - f(z_0)} ds \\ &= \frac{1}{2\pi i} \int_\gamma \frac{sf'(s)}{\frac{f(s)-f(z_0)}{s-z_0}} \frac{1}{s - z_0} ds \end{aligned}$$

Since $\lim_{s \to z_0} \frac{sf'(s)}{\frac{f(s)-f(z_0)}{s-z_0}} = z_0$, it has a removable singularity, hence we can apply Cauchy integral formula to get $g(f(z_0)) = z_0$. Note that we made no assumptions on the regularity of f^{-1}.

(6) Any conformal map can be obtained from another conformal map by composing conformal maps. We know that there is a conformal map from the open unit disk to the upper half plane. Taking a conformal map that maps from some disk of our choosing to the unit disk (since dilations and translations are conformal), then composing with the conformal map to the upper half plane, then taking another conformal map (rotations and translations) to some given half plane, gives us the required mapping.

(7) Using the fact that $2\cos(x) = e^{ix} + e^{-ix}$, we rewrite the integral as

$$\frac{1}{2}\int_0^{2\pi} e^{\cos\theta}(e^{in\theta - i\sin\theta} + e^{-in\theta + i\sin\theta})d\theta = \frac{1}{2}\int_0^{2\pi} e^{\cos\theta - i\sin\theta}e^{in\theta} + e^{\cos\theta + i\sin\theta}e^{-in\theta}d\theta.$$

Using the parametrization $z = e^{i\theta}$, the above becomes

$$\frac{1}{2i}\int_{|z|=1} e^{1/z}z^n + e^z z^{-n}dz$$

By residue theorem (using the series expansion for e^z), we compute that the above is $\frac{2\pi}{n!}$.